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FOURIER INVERSION OF INVARIANT INTEGRALS ON SEMISIMPLE REAL LIE GROUPS

BY

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ABSTRACT. Let G be a connected, semisimple real Lie group with finite center. Associated with every regular semisimple element g of G is a tempered invariant distribution Λ_g given by an orbital integral. This paper gives an inductive formula for computing the Fourier transform of Λ_g in terms of the space of tempered invariant eigendistributions of G.

1. Introduction. Let G be a connected semisimple real Lie group with finite center. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}_{\mathbf{C}}$ its complexification. If $G_{\mathbf{C}}$ is the simply connected complex analytic group corresponding to $\mathfrak{g}_{\mathbf{C}}$, we assume that G is the real analytic subgroup of $G_{\mathbf{C}}$ corresponding to $\mathfrak{g}$.

Let g be any regular semisimple element of G. Associated with g is the tempered invariant distribution Λ_g given by $f \mapsto \Lambda_g(f) = F_f^H(g), f \in C_c^\infty(G)$, where H is the unique Cartan subgroup of G containing g and F_f^H is the invariant integral of f relative to H defined by Harish-Chandra **[1(a)]**.

The purpose of this paper is to give explicit formulas for the Fourier transform of Λ_g. That is, we determine a linear functional $\hat{\Lambda}_g$ such that $\Lambda_g(f) = \hat{\Lambda}_g(\hat{f}), f \in C_c^\infty(G)$. Here $\hat{f}$ is defined on the space of tempered invariant eigendistributions of G which includes, for each conjugacy class $[H]$ of Cartan subgroups of G, a series of tempered invariant eigendistributions of G parameterized by the unitary character group $\hat{H}$ of H. These series include the characters of the discrete series representations of G (if G has a compact Cartan subgroup), the characters of the unitary principal series representations induced from each cuspidal parabolic subgroup of G, and certain "singular invariant eigendistributions" which can be interpreted as alternating sums of characters **[3]**.

The Fourier inversion formula was computed in the case that G has real rank one by P. Sally and G. Warner **[6]**, and in the case that G has real rank two by the author in her Ph. D. thesis **[2]**, written under the supervision of G. Warner. In the real rank one case, Sally and Warner use the inversion formula to compute the Plancherel formula for G. Also, D. Ragozin and G. Warner have recently used the inversion formula for real rank one groups, together with the Selberg trace formula, to obtain information on multiplici-

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ties of irreducible representations of G in $L^2(\Gamma \setminus G)$, Γ a co-compact discrete subgroup of G [**5**].

The method of computing the Fourier transform for general groups is an extension of that used by Sally and Warner in the rank one case. However, there are two problems in computing Fourier inversion formulas for groups G of real rank $n > 1$ which do not occur when $n = 1$, or when $n > 1$ but G has at most one conjugacy class of Cartan subgroups $[H]$ for each possible dimension of H_p, the vector part of H.

For $n = 1$, a Fourier inversion formula can be obtained for any regular semisimple element of G. The same is true when $n = 2$. However, there are cases when $n \geqslant 3$ (for example, Sp(3, **R**), the real symplectic group of rank three) in which certain integrals involved in computing the Fourier inversion formula for $F_f^H(h_0)$ diverge for certain regular elements $h_0 \in H'$. However, the Fourier inversion formula is still valid on a dense open set H^* of H.

A second problem which occurs when $n \geqslant 2$ is that in the final formula for the Fourier inversion of $F_f^H(h_0)$, the coefficients of $\theta(f)$, for some tempered invariant eigendistributions θ, are complicated expressions involving infinite sums which converge, but not absolutely, and have no obvious closed form. Thus they cannot be directly differentiated, term by term, to obtain a Plancherel formula for G. It is hoped, that in working with a specific group G of rank $\geqslant 2$, the Fourier inversion formulas can be greatly simplified.

In §2, the series of tempered invariant eigendistributions associated with each Cartan subgroup of G is described. In §3, Haar measures on G and its subgroups are normalized, the invariant integral is defined, and formulas which will be needed in §4 are listed. For convenience, for many definitions and results we refer to [**8**]. In §4 the main theorem of this paper, which can be used to obtain the complete inversion formula for $F_f^H(h_0)$, H any Cartan subgroup of G, $h_0 \in H^*$, is proved. In §5, the proof of a technical lemma from §2 is given.

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2. Tempered invariant eigendistributions on G. Let G and $\mathfrak{g}$ be as in §1. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ with Cartan involution θ. Let K be the maximal compact subgroup of G corresponding to $\mathfrak{k}$. Let H be a θ-stable Cartan subgroup of G with Lie algebra $\mathfrak{h}$. Then $\mathfrak{h}$ and H have decompositions $\mathfrak{h} = \mathfrak{h}_k + \mathfrak{h}_p$, $\mathfrak{h}_k = \mathfrak{h} \cap \mathfrak{k}$, $\mathfrak{h}_p = \mathfrak{h} \cap \mathfrak{p}$, and $H_K = H_K H_p$, $H_K = H \cap K$, $H_p = \exp(\mathfrak{h}_p)$.

Let $P \in \mathcal{P}(H_p)$, the set of parabolic subgroups of G with split part H_p. Let $P = MH_pN$ be the Langlands decomposition of P. Let $L = MH_p$. M and L are reductive subgroups of G, but need not be connected or acceptable, and

H_K is a compact Cartan subgroup of M. (If $H = H_K$ is a compact Cartan subgroup of G, then $P = M = G$.)

To each unitary character of H will be associated an invariant eigendistribution of G. These will include the characters of the principal series representations of G induced from P, along with certain "singular characters".

Let $L_H = \{\lambda \in \sqrt{-1}\, \mathfrak{h}_k^* | \xi_\lambda(\exp H) = \exp(\lambda(H)),\ H \in \mathfrak{h}_k$, extends to a well-defined character of H_K^0, the identity component of $H_K\}$. L_H is a lattice in $\sqrt{-1}\, \mathfrak{h}_k^*$. Corresponding to each $\lambda \in L_H$ there is an invariant eigendistribution $T(\lambda)$ defined on M^0, the identity component of M **[1(d), (e)]**. If $\lambda \in L'_H = \{\lambda \in L_H | \langle\lambda, \alpha\rangle \neq 0$ for all $\alpha \in \Phi(\mathfrak{m}_{\mathbf{C}}, \mathfrak{h}_{k\mathbf{C}})\}$ where $\Phi(\mathfrak{m}_{\mathbf{C}}, \mathfrak{h}_{k\mathbf{C}})$ denotes the set of roots of the complexified Lie algebra $\mathfrak{m}_{\mathbf{C}}$ of M with respect to $\mathfrak{h}_{k\mathbf{C}}$, then $T(\lambda)$ is, up to a sign, the character of a discrete series representation of M^0. If $\lambda \in L_H^s = L_H \setminus L'_H$, $T(\lambda)$ is an alternating sum of characters which are "limits of discrete series" **[3]**.

For any reductive group G and Cartan subgroup H, define $W(G, H) = N_G(H)/H$ where $N_G(H)$ is the normalizer of H in G. For G acceptable, define Δ_H^G as in **[8**, §8.1]. M and L need not be acceptable, but M^0 and L^0 are. If J is a Cartan subgroup of M or L we will write Δ_J^M and Δ_J^L with the understanding that these are well-defined on $J \cap M^0$ and $J \cap L^0$ respectively.

Let $H' = H \cap G'$ where G' denotes the set of regular elements of G. Then the formulas for $T(\lambda)$ on M^0 are given as follows. (See **[1(d)]**.)

$$T(\lambda)(h_k) = \Delta_{H_K}^M(h_k)^{-1} \sum_{w \in W(M^0, H_K^0)} \det w \xi_{w\lambda}(h_k), \qquad h_k \in H'_k \cap M^0. \tag{2.1}$$

Let $J = J_K J_p$ be a θ-stable Cartan subgroup of M^0. Pick a connected component J_I^+ of J_K. Let $k \in K \cap M^0$ satisfy $kJ_I^+k^{-1} \subseteq H_K^0$. Let $\mathfrak{z}$ denote the centralizer in $\mathfrak{m}$ of J_I^+, Z the connected subgroup of M^0 corresponding to $\mathfrak{z}$. Let $\mathfrak{j}'(\mathfrak{z}) = \{H \in \mathfrak{j}_p | \alpha(H) \neq 0$ for all $\alpha \in \Phi(\mathfrak{z}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}})\}$. Let $\mathfrak{j}_p^+$ be a connected component of $\mathfrak{j}'(\mathfrak{z})$, $J_p^+ = \exp(\mathfrak{j}_p^+)$. Let $y \in Z_{\mathbf{C}}$ satisfy $\mathrm{Ad}(yk^{-1})(\mathfrak{h}_{k\mathbf{C}}) = \mathfrak{j}_{\mathbf{C}}$. Let $j = j_k j_p \in J', j_k \in J_I^+, j_p = \exp(H_p) \in J_p^+$. Then

$$T(\lambda)(j_k j_p) = \Delta_J^M(j_k j_p)^{-1} \sum_{t \in W({}^kZ, H_K^0) \backslash W(M^0, H_K^0)} \det t \xi_{t\lambda}({}^k j_k)$$
$$\times \sum_{s \in W(Z, J)} \det s c_{\mathfrak{z}}(s\colon t\lambda\colon \mathfrak{j}_p^+) \exp\left(s^{yk^{-1}}(t\lambda)(H_p)\right). \tag{2.2}$$

The $c_{\mathfrak{z}}(s\colon \lambda\colon \mathfrak{j}_p^+)$ are constants satisfying:

$$c_{\mathfrak{z}}(s's\colon \lambda\colon s'\mathfrak{j}_p^+) = c_{\mathfrak{z}}(s\colon \lambda\colon \mathfrak{j}_p^+), \qquad s' \in W(Z, J); \tag{2.3}$$

$$c_{\mathfrak{z}}\big(s^{yk^{-1}}u:\lambda:\mathfrak{j}_p^+\big)=c_{\mathfrak{z}}(s:u\lambda:\mathfrak{j}_p^+),\qquad u\in W\big({}^kZ,H_K^0\big).\tag{2.4}$$

Fix J_I^+ and $\mathfrak{j}_p^+$. Let $F_{\mathfrak{z}}=\{{}^{yk^{-1}}\lambda|\lambda\in\sqrt{-1}\,\mathfrak{j}_k^*\}$ and $L_{\mathfrak{z}}={}^{yk^{-1}}L_H$. Set $F'_{\mathfrak{z}}=\{\lambda\in F_{\mathfrak{z}}|\langle\alpha,\lambda\rangle\neq 0$ for all $\alpha\in\Phi(\mathfrak{z}_C,\mathfrak{j}_C)\}$. $L'_{\mathfrak{z}}=L_{\mathfrak{z}}\cap F'_{\mathfrak{z}}$ is called the set of regular elements of $L_{\mathfrak{z}}$, $L^s_{\mathfrak{z}}=L_{\mathfrak{z}}\setminus L'_{\mathfrak{z}}$ the set of singular elements. For $\lambda\in L_{\mathfrak{z}}$, we will write $c(s:\lambda)$ for $c_{\mathfrak{z}}(s:{}^{ky^{-1}}\lambda:\mathfrak{j}_p^+)$. Then for $\lambda\in L'_{\mathfrak{z}}$, the constants $c(s:\lambda)$ are uniquely determined since the terms $\exp(s\lambda)$ are linearly independent on J' for $s\in W(Z,J)$. They depend only on the connected component F^+ of $\lambda\in F'_{\mathfrak{z}}$. Write $c(s:\lambda)=c(s:F^+)$ when $\lambda\in F^+$.

If $\lambda\in L^s_{\mathfrak{z}}$, there exists some $s\neq 1$ in $W(Z,J)$ for which $s\lambda=\lambda$. Thus the constants $c(s:\lambda)$ are not uniquely determined. Let $F_1,\dots,F_k$ be the connected components of $F'_{\mathfrak{z}}$ with $\lambda\in\bar{F}_i$, $i=1,\dots,k$. Then Harish-Chandra defines the constants $c(s:\lambda)$ by the average,

$$c(s:\lambda)=\frac{1}{k}\sum_{i=1}^{k}c(s:F_i)$$

[1(e)]. In order to prove convergence of certain integrals in §4, it is necessary to make a different, but equivalent, definition for certain singular λ.

Let $\Phi^+=\{\alpha\in\Phi(\mathfrak{z}_C,\mathfrak{j}_C)|\alpha(H)>0$ for all $H\in\mathfrak{j}_p^+\}$. Let $\{\alpha_1,\dots,\alpha_l\}$ be the set of simple roots for Φ^+ and $\{H_1,\dots,H_l\}$ the dual basis of $\mathfrak{j}_p$ defined by $\alpha_i(H_j)=\delta_{ij}$. Then $\mathfrak{j}_p^+=\{\sum_{i=1}^l r_iH_i|r_i>0,\ i=1,\dots,l\}$. For $\lambda\in L_{\mathfrak{z}}$, $s\in W(Z,J)$, write $s\lambda=\lambda_I(s\lambda)+\sum_{i=1}^l t_i(s\lambda)\alpha_i$ where $\lambda_I(s\lambda)\in\mathfrak{j}_C^*$ assumes purely imaginary values on $\mathfrak{j}$, and $t_i(s\lambda)$ are real numbers, $1\leqslant i\leqslant l$. Then $c(s:\lambda)=0$ if $\mathrm{Re}(s\lambda(H))>0$ for any $H\in\mathfrak{j}_p^+$ **[1(d)]**, so $c(s:\lambda)=0$ if $t_i(s\lambda)>0$ for any $1\leqslant i\leqslant l$.

LEMMA 2.5. *Fix $\lambda_0\in L_{\mathfrak{z}}$, $s\in W(Z,J)$, and suppose $t_i(s\lambda)=0$ for some $1\leqslant i\leqslant l$. Let $W'=\{w\in W(Z,J)|ws\lambda_0=s\lambda_0\}$. Then*

$$\sum_{w\in W'}\det w\,c(ws:\lambda_0)=0.$$

Since the proof of Lemma 2.5 is long and technical we defer it to §5.

COROLLARY 2.6. *Suppose $t_i(s\lambda)=0$ for any $1\leqslant i\leqslant l$. Then if λ is regular, $c(s:\lambda)=0$, and if λ is singular, we may redefine $c(s:\lambda)=0$ without changing any of the properties of the constants $c(s:\lambda)$.*

For $\lambda\in L'_H$, let $\varepsilon(\lambda)=\operatorname{sign}\prod_{\alpha\in\Phi^+(\mathfrak{m}_C,\mathfrak{h}_{kC})}\langle\alpha,\lambda\rangle$. Let $q=\frac{1}{2}\dim(M/M\cap K)$. Then for $\lambda\in L'_H$, $(-1)^q\varepsilon(\lambda)T(\lambda)$ is the character of a representation of the discrete series of M^0. For $\lambda\in L^s_H$, if $w\lambda=\lambda$ for $w\neq 1$ in $W(M^0,H_K^0)$, then $T(\lambda)=0$. Otherwise, $T(\lambda)$ is the alternating sum of characters which can be explicitly embedded in a certain reducible unitary principal series representation of M^0 **[3]**.

Let $Z(\mathfrak{h}_p) = K \cap \exp(\sqrt{-1}\,\mathfrak{h}_p)$. This is a finite Abelian group generated by elements of order two, and $H_K = Z(\mathfrak{h}_p)H_K^0$. Define $M^+ = Z(\mathfrak{h}_p)M^0$. Let $\rho(\mathfrak{m}, \mathfrak{h}_k) = \frac{1}{2}\Sigma\alpha$, $\alpha \in \Phi^+(\mathfrak{m}_C, \mathfrak{h}_{kC})$, and $\Gamma_0 = Z(\mathfrak{h}_p) \cap H_K^0$. Let $\tilde{H}_K$ denote the set of all pairs $\lambda \in L_H$ and $\eta \in Z(\mathfrak{h}_p)\hat{}$ such that $\eta|\Gamma_0 = \xi_{\lambda - \rho(\mathfrak{m},\mathfrak{h}_k)}|\Gamma_0$. Then given $(\lambda, \eta) \in \tilde{H}_K$ there is an invariant eigendistribution $T(\lambda, \eta)$ on M with support on M^+ given by:

$$T(\lambda, \eta)(zm) = \eta(z) \sum_{\gamma \in M/M^+} T(\lambda)(\gamma m \gamma^{-1}),$$

$$z \in Z(\mathfrak{h}_p), \quad m \in M^0. \qquad (2.7)$$

(See [9].)

For any Cartan subgroup J of M, representatives γ for the cosets of M/M^+ may be chosen so that γ normalizes J and centralizes J_p. In particular, for $h_k \in H_K'$, $T(\lambda, \eta)$ has the simple formula

$$T(\lambda, \eta)(zh_k) = \eta(z)\Delta^M_{H_K}(h_k)^{-1} \sum_{w \in W(M, H_K)} \det w\xi_{w\lambda}(h_k),$$

$$z \in Z(\mathfrak{h}_p), h_k \in H_K^0. \qquad (2.8)$$

This formula is obtained from (2.7) using the fact that $W(M, H_K)$ is generated by $W(M^0, H_K^0)$ and conjugation by certain representatives of the cosets of M/M^+.

There is a one-to-one correspondence between pairs $(\lambda, \eta) \in \tilde{H}_K$ and elements $b^* \in \hat{H}_K$ given as follows. Elements of $\hat{H}_K$ are of the form $\chi \otimes \xi_\lambda$ where $\chi \in Z(\mathfrak{h}_p)\hat{}$, $\lambda \in L_H$, and $\chi|\Gamma_0 = \xi_\lambda|\Gamma_0$. Let $\xi_H = \xi_{\rho(\mathfrak{g},\mathfrak{h})}$, $\rho(\mathfrak{g}, \mathfrak{h}) = \frac{1}{2}\Sigma\alpha$, $\alpha \in \Phi^+(\mathfrak{g}_C, \mathfrak{h}_C)$. If $\chi \otimes \xi_\lambda \in \hat{H}_K$, then $(\lambda, \xi_H^{-1} \otimes \chi) \in \tilde{H}_K$ as $\xi_H^{-1} \otimes \chi|\Gamma_0 = \xi_{\lambda - \rho(\mathfrak{g},\mathfrak{h})}|\Gamma_0 = \xi_{\lambda - \rho(\mathfrak{m},\mathfrak{h}_k)}|\Gamma_0$. Conversely, if $(\lambda, \eta) \in \tilde{H}_K$, then $\chi = \eta \otimes \xi_H \in Z(\mathfrak{h}_p)\hat{}$ and $\chi \otimes \xi_\lambda \in \hat{H}_K$. For $b^* = \chi \otimes \xi_\lambda \in \hat{H}_K$, we write $T(b^*)$ for $T(\lambda, \xi_H^{-1} \otimes \chi)$.

Each character of H_p is of the form $h_p^{\sqrt{(-1)}\nu} = \exp(\sqrt{-1}\,\nu \log h_p)$, $h_p \in H_p$, $\nu \in \mathfrak{h}_p^*$, where log: $H_p \to \mathfrak{h}_p$ denotes the inverse of exp: $\mathfrak{h}_p \to H_p$. Given $(\lambda, \eta) \in \tilde{H}_K$ and $\nu \in \mathfrak{h}_p^*$, there is an invariant eigendistribution $\theta(H, \lambda, \eta, \nu)$ on G with support on the closure of $\bigcup G^J$, J a Cartan subgroup of L, where $G^J = \bigcup_{x \in G} xJ'x^{-1}$.

For any Cartan subgroup J of L, let $J = J_1, J_2, \ldots, J_k$ be a complete set of representatives for distinct conjugacy classes of Cartan subgroups of L for which J_i is conjugate to J in G, $i = 1, \ldots, k$. (If M is acceptable, k is always one.) Let $x_i \in G$ satisfy $J_i = x_iJx_i^{-1}$ and for $j \in J$, write $j_i = x_ijx_i^{-1}$. Then for $j \in J'$,

$$\theta(H,\lambda,\eta,\nu)(j)=\sum_{i=1}^{k}\left[W(L,J_i)\right]^{-1}\left|\Delta^G_{J_i}(j_i)\right|^{-1}$$
$$\times\sum_{w\in W(G,J_i)}\left|\Delta^L_{J_i}(wj_i)\right|T(\lambda,\eta)(wj_i|M)(wj_i|H_p)^{\vee(-1)\nu},\quad(2.9)$$

where for $j \in J_i = (J_i \cap M)H_p$, $j|M$ and $j|H_p$ denote the components of j in $J_i \cap M$ and H_p respectively (see [7]). Note that $\Delta^L_{J_i}$ is defined on $J_i \cap L^0$ only. However $|\Delta^L_{J_i}|$ makes sense on all of J_i. If $j \in J_i, j = zj^0, z \in Z(\mathfrak{h}_p), j^0 \in J_i \cap L^0$, $|\Delta^L_{J_i}(zj^0)|$ is defined to be $|\Delta^L_{J_i}(j^0)|$. This is well defined as $\Delta^L_{J_i}(zj^0) = \pm\Delta^L_{J_i}(j^0)$ for $z \in Z(\mathfrak{h}_p) \cap L^0$. For $b^* = \chi \otimes \xi_\lambda \in \hat{H}_K$, $\nu \in \mathfrak{h}_p^*$, write $\theta(H, b^*, \nu)$ for $\theta(H, \lambda, \xi_H^{-1} \otimes \chi, \nu)$.

When $\lambda \in L'_H$, $(-1)^q\varepsilon(\lambda)\theta(H, \lambda, \eta, \nu)$ is the character of a representation of the unitary principal series of G induced from P. If $\lambda \in L^s_H$, $\theta(H, \lambda, \eta, \nu)$ is zero, or is an alternating sum of characters which can be embedded in a reducible unitary principal series representation associated to a different cuspidal parabolic subgroup [3].

3. The invariant integral. We must first normalize various invariant measures on G. For each Cartan subgroup H of G, let $x \to \dot{x}$ denote the canonical projection of G onto G/H. Normalize the G-invariant measure $d_{G/H}(\dot{x})$ on G/H as in [8, §8.1.2]. Let H^1 be a fundamental θ-stable Cartan subgroup of G with Cartan subalgebra $\mathfrak{h}^1$. Write $H^1 = H^1_K H^1_p$. Normalize Haar measure on H^1 as follows. If $h = h_k h_p \in H^1$, $h_k \in H^1_K$, $h_p \in H^1_p$, let $d_{H^1}(h) = d_{H^1_K}(h_k)d_{H^1_p}(h_p)$, $d_{H^1_K}$ the Haar measure on H^1_K assigning it total mass one, $d_{H^1_p}$ the Haar measure on H^1_p which is the transport via the exponential map of the canonical Haar measure on $\mathfrak{h}^1_p$ associated with the Euclidean structure derived from the Killing form on G. Then a Haar measure on G is fixed by

$$\int_G f(x)d_G(x)=\int_{G/H^1}\int_{H^1} f(xh)d_{H^1}(h)d_{G/H^1}(\dot{x})$$

for all $f \in C_c(G)$.

A Haar measure on each θ-stable Cartan subgroup H of G is then fixed by requiring that for all $f \in C_c(G)$,

$$\int_G f(x)d_G(x)=\int_{G/H}\int_{H} f(xh)d_{H}(h)d_{G/H}(\dot{x}).$$

Write $H = H_K H_p$. Let d_{H_p} be the Haar measure on H_p which is the transport via the exponential map of the canonical Haar measure on its Lie algebra $\mathfrak{h}_p$. Normalize Haar measure on H_K so that $d_H(h_K h_p) = d_{H_K}(h_K)d_{H_p}(h_p)$, $h_K \in H_K$, $h_p \in H_p$. Let $\mathrm{vol}(H_K)$ denote the total mass of H_K.

For any θ-stable Cartan subgroup H of G, $h = h_K h_p \in H'$, $w \in W(G, H)$, define $\varepsilon_R^H(h)$, $\varepsilon_R^H(w)$, and $\varepsilon_W^H(h_k)$ as in [8, §8.1.1]. For $f \in C_c^\infty(G)$, $h \in H'$, the invariant integral of f relative to H is defined by:

$$F_f^H(h) = \varepsilon_R^H(h)\Delta_H^G(h)\int_{G/H} f(xhx^{-1})\, d_{G/H}(\dot{x}). \tag{3.1}$$

Then $F_f^H \in L^1(H)$ and is C^∞ on H'. Further, there is a compact subset of H off which F_f^H vanishes. Let $G^H = \bigcup_{x \in G} xH'x^{-1}$. Then from Weyl's formula it follows that

$$\int_{G^H} f(x) d_G(x) = [W(G,H)]^{-1}\int_H \varepsilon_R^H(h)\,\overline{\Delta_H^G(h)}\,F_f^H(h) d_H(h),$$

$$f \in C_c^\infty(G). \tag{3.2}$$

For $b^* \in \hat{H}_K$, $\nu \in \mathfrak{h}_p^*$, we define the Fourier transform of F_f^H at (b^*, ν) by

$$\hat{F}_f^H(b^*, \nu) = (2\pi)^{-\dim \mathfrak{h}_p/2}\int_{H_K \times H_p} F_f^H(h_k h_p) b^*(h_k) h_p^{\sqrt{(-1)}\nu} d_{H_K}(h_k) d_{H_p}(h_p). \tag{3.3}$$

Then since $F_f^H \in C^\infty(H')$, for $h = h_k h_p \in H'$ we have

$$F_f^H(h_k h_p) = (2\pi)^{-\dim \mathfrak{h}_p/2}(\operatorname{vol} H_K)^{-1} \sum_{b^* \in \hat{H}_K} \overline{b^*(h_k)} \int_{\mathfrak{h}_p^*} \hat{F}_f^H(b^*, \nu) h_p^{-\sqrt{(-1)}\nu}\, d\nu \tag{3.4}$$

for a suitable normalization of Haar measure on $\mathfrak{h}_p^*$ and a suitable definition $\Sigma_{b^* \in \hat{H}_K}$ which will be made later.

We will need the following facts taken from [8, Chapter 8]. Let $w \in W(G, H)$, $h = h_k h_p \in H'$. Define M corresponding to H as in §2. Then:

$$\Delta_H^G(wh) = \det w \Delta_H^G(h); \tag{3.5}$$

$$\varepsilon_R^H(wh) = \varepsilon_R^H(w)\varepsilon_w^H(h_k)\varepsilon_R^H(h); \tag{3.6}$$

$$F_f^H(wh) = \det w \varepsilon_R^H(w)\varepsilon_w^H(h_k)F_f^H(h); \tag{3.7}$$

$$\overline{\Delta_H^G(h)} = (-1)^{r_I(H)}\Delta_H^G(h), \tag{3.8}$$

where $r_I(H)$ is the number of elements of $\Phi^+(\mathfrak{g}_\mathbf{C}, \mathfrak{h}_\mathbf{C})$ taking purely imaginary values on $\mathfrak{h}$.

Let H_I^+ be a connected component of H_K, $\mathfrak{z}$ the centralizer in $\mathfrak{g}$ of H_I^+. Let $\Phi^+(\mathfrak{z}_\mathbf{C}, \mathfrak{h}_\mathbf{C}) = \Phi^+(\mathfrak{g}_\mathbf{C}, \mathfrak{h}_\mathbf{C}) \cap \Phi(\mathfrak{z}_\mathbf{C}, \mathfrak{h}_\mathbf{C})$, and let $\mathfrak{h}_p^+ = \{H \in \mathfrak{h}_p | \alpha(H) > 0$ for all $\alpha \in \Phi^+(\mathfrak{z}_\mathbf{C}, \mathfrak{h}_\mathbf{C})\}$. Then

$$\varepsilon_R^H(h_k h_p) = 1 \qquad \text{for } h_k \in H_I^+, h_p \in \mathfrak{h}_p^+. \tag{3.9}$$

Fix an ordering on the roots of $\Phi(\mathfrak{g}_\mathbf{C}, \mathfrak{h}_\mathbf{C})$ so that for $\alpha \in \Phi(\mathfrak{g}_\mathbf{C}, \mathfrak{h}_\mathbf{C})$ which does not take purely imaginary values on $\mathfrak{h}$, α is positive if and only if α^σ is positive. Hence $\alpha^\sigma(H) = \alpha(\sigma(H))$ for all $H \in \mathfrak{h}_\mathbf{C}$, and σ denotes the conjugation of $\mathfrak{g}_C$ with respect to $\mathfrak{g}$. Let $\Phi^+(\mathfrak{m}_C, \mathfrak{h}_{kC}) = \Phi^+(\mathfrak{g}_C, \mathfrak{h}_C) \cap \Phi(\mathfrak{m}_C, h_{kC})$. Then with respect to these orderings,

$$\operatorname{sign}\left\{ \frac{\Delta_H^G(\gamma h_k h_p)}{\Delta_{H_K}^M(h_k)} \right\} = \varepsilon_R^H(\gamma h_k h_p)\xi_H(\gamma)$$

$$\text{for } \gamma \in Z(\mathfrak{h}_p),\ h_k \in (H_K^0)',\ h_p \in H_p. \qquad (3.10)$$

Here $\xi_H = \xi_{\rho(\mathfrak{g},\mathfrak{h})}$ as in §2. Further, if J is any Cartan subgroup of L, $J_M = J \cap M$, then for $\gamma \in Z(\mathfrak{h}_p), j_M \in J' \cap M^0, h_p \in H_p$,

$$\operatorname{sign}\left\{ \frac{\Delta_J^G(\gamma j_M h_p)}{\Delta_{J_M}^M(j_M)} \right\} = \varepsilon_R^J(\gamma j_M h_p)\varepsilon_R^{J_M}(j_M)\xi_J(\gamma). \qquad (3.11)$$

Here $\xi_J = \xi_{\rho(\mathfrak{g},\mathfrak{j})}$ and $\varepsilon_R^{J_M}$ is defined with respect to M and J_M. Again, $\Phi^+(\mathfrak{g}_C, \mathfrak{j}_C)$ is chosen so that for $\alpha \in \Phi(\mathfrak{g}_C, \mathfrak{j}_C)$, α not taking purely imaginary values on $\mathfrak{j}$, α is positive if and only if α^σ is positive. $\Phi^+(\mathfrak{m}_C, \mathfrak{j}_{M_C}) = \Phi^+(\mathfrak{g}_C, \mathfrak{j}_C) \cap \Phi(\mathfrak{m}_C, \mathfrak{j}_{M_C})$.

4. The Fourier inversion formula. Fix a θ-stable Cartan subgroup H of G. Let M be as in §2. Let $\operatorname{Car}(G, H)$ be a complete set of θ-stable representatives for the G-conjugacy classes of Cartan subgroups of $L = MH_p$. Let $\operatorname{Car}'(G, H) = \operatorname{Car}(G, H) \setminus \{H\}$. Let ξ_H and $r_I(H)$ be as defined in (3.10) and (3.8) respectively. Set $d(H_p) = \dim \mathfrak{h}_p$.

LEMMA 4.1. *Let $f \in C_c^\infty(G)$, $b^* = \chi \otimes \xi_\lambda \in \hat{H}_K$, $\nu \in \mathfrak{h}_p^*$. Then*

$$\hat{F}_f^H(b^*, \nu)$$

$$= \frac{(-1)^{r_I(H)}}{(2\pi)^{d(H_p)/2}} \Big\{ \Theta(H, \lambda, \chi \otimes \xi_H^{-1}, \nu)(f)$$

$$- \sum_{J \in \operatorname{Car}'(G, H)} \int_{G^J} \Theta(H, \lambda, \chi \otimes \xi_H^{-1}, \nu)(x) f(x) d_G(x) \Big\}.$$

PROOF. Since $\theta = \theta(H, \lambda, \chi \otimes \xi_H^{-1}, \nu)$ has support on $\bigcup_{J \in \operatorname{Car}(G, H)} G^J$,

$$\theta(f) = \int_G \theta(x) f(x) d_G(x) = \sum_{J \in \operatorname{Car}(G, H)} \int_{G^J} \theta(x) f(x) d_G(x).$$

Using (3.2), (3.8), (2.9), (3.10), (2.8), and (3.3) respectively, we have

$$\int_{G^H}\theta(x)f(x)d_G(x) = [W(G,H)]^{-1}\int_H \varepsilon_R^H(h)\overline{\Delta_H^G(h)}F_f^H(h)\theta(h)d_H(h)$$

$$= (-1)^{r_I(H)}[W(M,H_K)]^{-1}[W(G,H)]^{-1}\sum_{w\in W(G,H)}$$

$$\times\int_{H_K\times H_p}\varepsilon_R^H(h_kh_p)\Delta_H^G(h_kh_p)F_f^H(h_kh_p)\frac{|\Delta_{H_K}^M(h_k)|}{|\Delta_H^G(h_kh_p)|}$$

$$\times T(\lambda,\chi\otimes\xi_H^{-1})(wh_k)(wh_p)^{\sqrt{(-1)}\nu}d_{H_K}(h_k)d_{H_p}(h_p)$$

$$= (-1)^{r_I(H)}[W(M,H_K)]^{-1}\sum_{\gamma\in Z'}\int_{H_K^0\times H_p}F_f^H(\gamma h_kh_p)$$

$$\times\left(\sum_{w\in W(M,H_K)}\det w\xi_\lambda(wh_k)\right)\chi(\gamma)h_p^{\sqrt{(-1)}\nu}d_{H_k}(h_k)d_{H_p}(h_p)$$

$$= (-1)^{r_I(H)}(2\pi)^{d(H_p)/2}\hat{F}_f^H(b^*,\nu),$$

where $Z' = Z(\mathfrak{h}_p)/Z(\mathfrak{h}_p)\cap H_K^0$. Q.E.D.

In order to consider the Fourier series of F_f^H, we must make precise the type of convergence used relative to $\hat{H}_K$. Let $\{\beta_1,\ldots,\beta_r\}$ be the set of simple roots of $\Phi^+(\mathfrak{m}_{\mathbf{C}},\mathfrak{h}_{k\mathbf{C}})$. Let $\{\Lambda_1,\ldots,\Lambda_r\}$ be the basis of $\sqrt{-1}\,\mathfrak{h}_k^*$, satisfying $2\langle\Lambda_i,\beta_j\rangle/\langle\beta_j,\beta_j\rangle = \delta_{ij}$, $1 \leqslant i,j \leqslant r$. Then $L_H = \{\sum_{i=1}^r m_i\Lambda_i | m_i \in Z\}$. For any positive integer M, define $L_H^M = \{\sum_{i=1}^r m_i\Lambda_i | -M \leqslant m_i \leqslant M, 1 \leqslant i \leqslant r\}$. Given $\chi \in Z(\mathfrak{h}_p)\hat{}$, let $L_\chi = \{\lambda \in L_H | \xi_\lambda|_{\Gamma_0} = \chi|_{\Gamma_0}\}$, $\Gamma_0 = Z(\mathfrak{h}_p)\cap H_K^0$. Let $L_\chi^M = L_\chi \cap L_H^M$. Then summability relative to H_K is defined by

$$\sum_{b^*\in\hat{H}_K} = \lim_{M\to\infty}\sum_{\chi\in Z(\mathfrak{h}_p)\hat{}}\ \sum_{\lambda\in L_\chi^M}.$$

LEMMA 4.2. *Let $h_0 = h_kh_p \in H'$. Then*

$$F_f^H(h_0) = \frac{(-1)^{r_I(H)}}{\operatorname{vol} H_K(2\pi)^{d(H_p)}}\sum_{b^*\in\hat{H}_K}\overline{b^*(h_k)}$$

$$\times\int_{\mathfrak{h}_p^*}h_p^{-\sqrt{(-1)}\nu}\theta(H,b^*,\nu)(f)\,d\nu + I_f^H(h_0),$$

where

$$I_f^H(h_0) = \frac{(-1)^{r_I(H)+1}}{\operatorname{vol} H_K(2\pi)^{d(H_p)}}\sum_{b^*\in\hat{H}_K}\overline{b^*(h_k)}\int_{\mathfrak{h}_p^*}h_p^{-\sqrt{(-1)}\nu}$$

$$\times\left\{\sum_{J\in\operatorname{Car}'(G,H)}\int_{G^J}\theta(H,b^*,\nu)(x)f(x)d_G(x)\right\}d\nu.$$

PROOF. Use (3.4) and Lemma 4.1. Since

$$\sum_{b^* \in \hat{H}_K} \overline{b^*(h_k)} \int_{\mathfrak{h}_p^*} h^{-\sqrt{(-1)}\nu} \theta(H, b^*, \nu)(f)\, d\nu$$

converges absolutely the result is clear [1(d)], [9]. Q.E.D.

The remainder of §4 is devoted to an analysis of $I_f^H(h_0)$. The main result of this paper is that there is a dense open set $H^* \subseteq H'$ such that if $h_0 \in H^*$, then for $J = J_K J_p \in \mathrm{Car}'(L, H)$ (see Lemma 4.3),

$$\sum_{b^* \in \hat{H}_K} \overline{b^*(h_k)} \int_{\mathfrak{h}_p^*} h_p^{-\sqrt{(-1)}\nu} \int_{G^J} \theta(H, b^*, \nu)(x) f(x) d_G(x)\, d\nu$$

can be expressed as a finite number of terms of the form

$$\sum_{b^* \in \hat{J}_K} \overline{b^*(j_k)} \int_{\mathfrak{j}_p^*} g(j_p, \mu) \hat{F}_f^J(b^*, \mu)\, d\mu$$

where $j_0 = j_k j_p$ depends on $h_0 \in H^*$ and $g(j_p, \mu)$ is a continuous function of j_p and μ.

Let $Q = M_1 J_p N_1 \in \mathcal{P}(J_p)$. If J is a Cartan subgroup of G with $\dim J_p$ maximal, then Q is a minimal parabolic subgroup of G, M_1 is compact, and J_K is, up to conjugacy, its only Cartan subgroup. Thus

$$\hat{F}_f^J(b^*, \mu) = \frac{(-1)^{r_I(J)}}{(2\pi)^{d(J_p)/2}} \theta(J, b^*, \mu)(f)$$

is, up to scalar multiple, the character of a unitary principal series representation of G induced from the minimal parabolic subgroup Q.

Otherwise, if $\dim J_p$ is not maximal, let $\mathrm{Car}(G, J)$ be a complete set of θ-stable representatives for G-conjugacy classes of Cartan subgroups of $L_1 = M_1 J_p$, $\mathrm{Car}'(G, J) = \mathrm{Car}(G, J) \setminus \{J\}$. In this case, by applying Lemma (4.1) to J,

$$\hat{F}_f^J(b^*, \mu) = \frac{(-1)^{r_I(J)}}{(2\pi)^{d(J_p)/2}} \left\{ \theta(J, b^*, \mu)(f) - \sum_{A \in \mathrm{Car}'(G, J)} \int_{G^A} \theta(J, b^*, \mu)(x) f(x) d_G(x) \right\},$$

and so, as in Lemma 4.2,

$$\sum_{b^* \in \hat{J}_K} \overline{b^*(j_k)} \int_{\mathfrak{j}_p^*} g(j_p, \mu) \hat{F}_f^J(b^*, \mu)\, d\mu$$

$$= \frac{(-1)^{r_I(J)}}{(2\pi)^{d(J_p)/2}} \sum_{b^* \in \hat{J}_K} \overline{b^*(j_k)} \int_{\mathfrak{j}_p^*} g(j_p, \mu) \theta(J, b^*, \mu)(f)\, d\mu + I_f^J(j_0),$$

where

$$I_f^J(j_0) = \frac{(-1)^{r_I(J)+1}}{(2\pi)^{d(J_p)/2}} \sum_{b^* \in \hat{J}_K} \overline{b^*(j_k)}$$

$$\times \int_{\mathfrak{j}_p^*} g(j_p, \mu) \left\{ \sum_{A \in \mathrm{Car}'(G, J)} \int_{G^A} \theta(J, b^*, \mu)(x) f(x) d_G(x) \right\} d\mu.$$

Clearly each of the terms $I_f^J(j_0)$ can be evaluated in the same manner as $I_f^H(h_0)$. Further, $I_f^H(h_0)$ involved terms corresponding to Cartan subgroups with $\dim J_p > \dim H_p$. Each $I_f^J(j_0)$ involves only terms corresponding to Cartan subgroups $A \in \mathrm{Car}'(G, J)$ for which $\dim A_p > \dim J_p$. Thus the process will terminate in a finite number of steps.

Fix a Cartan subgroup J of L. Write $J = J_M H_p$. $J_M = J \cap M$ is a Cartan subgroup of M with Cartan decomposition $J_M = J_K J_{M_p}$. Write $A = J_{M_p}$. Define d_A on A to be the transport via the exponential map of the canonical Haar measure on $\mathfrak{a}$, the Lie algebra of A. Then since $\mathfrak{j}_p = \mathfrak{h}_p + \mathfrak{a}$ is an orthogonal direct sum, $d_{J_p}(h_1 h_2) = d_A(h_1) d_{H_p}(h_2)$, $h_1 \in A$, $h_p \in H_p$.

Let Γ_1 be a set of representatives in $Z(\mathfrak{h}_p)$ for $Z(\mathfrak{h}_p)/Z(\mathfrak{h}_p) \cap J_K^0$. Let Γ_2 be a set of representatives in $Z(\mathfrak{a})$ for $Z(\mathfrak{a})/Z(\mathfrak{a}) \cap Z(\mathfrak{h}_p)J_K^0$. Let Γ_3 be a set of representatives for M/M^+ which normalize J_M and centralize J_p.

Note that $Z(\mathfrak{h}_p)$ and $Z(\mathfrak{a})$ are subgroups of the abelian group $Z(\mathfrak{j}_p)$ and so elements of Γ_1, Γ_2 and Γ_3 commute. Normalize Haar measure on J_K^0 so that for $f \in C(J_K \cap M^+)$,

$$\int_{J_K \cap M^+} f(j_k) d_{J_K}(j_k) = \sum_{\gamma_1 \in \Gamma_1} \sum_{\gamma_2 \in \Gamma_2} \int_{J_K^0} f(\gamma_1 \gamma_2 j_k) d_{J_K^0}(j_k).$$

We will simplify notation by always writing $j = \gamma_1 \gamma_2 j_k h_1 h_2$ for $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$, $j_k \in J_K^0$, $h_1 \in A$ and $h_2 \in H_p$, and abbreviating $d_{J_K^0}(j_k)$, $d_A(h_1)$, and $d_{H_p}(h_2)$ by dj_k, dh_1, and dh_2 respectively.

LEMMA 4.3. *Let* $\mathrm{Car}(L, H)$ *be a set of representatives for the L-conjugacy classes of Cartan subgroups of* L, $\mathrm{Car}'(L, H) = \mathrm{Car}(L, H) \setminus \{H\}$. *Then*

$$I_f^H(h_0) = \frac{(-1)^{r_I(H)+1}[\Gamma_3]}{\mathrm{vol}\, H_K (2\pi)^{d(H_p)}} \sum_{b^* \in \hat{H}_K} \overline{b^*(h_k)} \int_{\mathfrak{h}_p^*} h_p^{-\sqrt{(-1)}\nu}$$

$$\times \left\{ \sum_{J \in \mathrm{Car}'(L, H)} \frac{(-1)^{r_I(J)}}{[W(M, J_M)]} \sum_{\gamma_1 \in \Gamma_1} \sum_{\gamma_2 \in \Gamma_2} I_f^H(J, b^*, \nu, \gamma_1, \gamma_2) \right\} d\nu$$

where

$$I_f^H(J, b^*, \nu, \gamma_1, \gamma_2) = \int_{J_K^0 \times A \times H_p} \varepsilon_R^{J_M}(\gamma_2 j_k h_1) F_f^J(\gamma_1\gamma_2 j_k h_1 h_2)\Delta_{J_M}^M(\gamma_2 j_k h_1)$$

$$\times \chi(\gamma_1) T(\lambda)(\gamma_2 j_k h_1) h_2^{\vee(-1)\nu}\, dj_k\, dh_1\, dh_2.$$

PROOF. Using (3.2), (2.9), (3.5), and (3.11),

$$\sum_{J \in \mathrm{Car}'(G,H)} \int_{G^J} \theta(H, b^*, \nu)(x) f(x) d_G(x)$$

$$= \sum_{J \in \mathrm{Car}'(G,H)} \frac{(-1)^{r_I(J)}}{[W(G,J)]} \int_J \varepsilon_R^J(j)\Delta_J^G(j) F_f^J(j)\theta(H, b^*, \nu)(j)\, dJ(j)$$

$$= \sum_{J \in \mathrm{Car}'(L,H)} \frac{(-1)^{r_I(J)}}{[W(G,J)]} \sum_{w \in W(G,J)} \int_J \varepsilon_R^J(j) \frac{\Delta_J^G(j)}{|\Delta_J^G(j)|}$$

$$\times F_f^J(j) \frac{|\Delta_J^L(wj)|}{[W(L,J)]} T(b^*)(wj|M)(wj|H_p)^{\vee(-1)\nu}\, dj$$

$$= \sum_{J \in \mathrm{Car}'(L,H)} \frac{(-1)^{r_I(J)}}{[W(M,J_M)]} \sum_{\gamma_1 \in \Gamma_1} \sum_{\gamma_2 \in \Gamma_2}$$

$$\times \int_{J_K^0 \times A \times H_p} \varepsilon_R^{J_M}(\gamma_2 j_k h_1) F_f^J(\gamma_1\gamma_2 j_k h_1 h_2)\Delta_{J_M}^M(\gamma_2 j_k h_1)$$

$$\times \chi(\gamma_1) h_2^{\vee(-1)\nu} \sum_{\gamma_3 \in \Gamma_3} T(\lambda)(\gamma_3\gamma_2 j_k h_1 \gamma_3^{-1})\, dj_k\, dh_1\, dh_2$$

$$= \sum_{J \in \mathrm{Car}'(L,H)} \frac{(-1)^{r_I(J)}[\Gamma_3]}{[W(M,J_M)]} \sum_{\gamma_1 \in \Gamma_1} \sum_{\gamma_2 \in \Gamma_2} I_f^H(J, b^*, \nu, \gamma_1, \gamma_2).$$

To expand the sum over $\mathrm{Car}'(G,H)$ to a sum over $\mathrm{Car}'(L,H)$ we use (2.9) and the fact that for $J_i = x_i J x_i^{-1}$, $x_i \in G$, $j_i = x_i j x_i^{-1}$ for $j \in J$, $\varepsilon_R^{J_i}(j_i) = \varepsilon_R^J(j)$, $\Delta_{J_i}^G(j_i) = \Delta_J^G(j)$, and $F_f^{J_i}(j_i) = F_f^J(j)$. Q.E.D.

Fix $J \in \mathrm{Car}'(L,H)$, $b^* = \xi_\lambda \otimes \chi \in \hat{H}_K$, $\nu \in \mathfrak{h}_p^*$, $\gamma_1 \in \Gamma_1$, and $\gamma_2 \in \Gamma_2$. Write $I(\gamma_1, \gamma_2)$ for $I_f^H(J, b^*, \nu, \gamma_1, \gamma_2)$. Define $\mathfrak{z}$, Z, k, and y as in §2 with respect to $J_I^+ = \gamma_2 J_K^0$. Use $\mathfrak{a}^*$ to denote the component of h_1 in $i'_M(\mathfrak{z})$. Let $\mathfrak{a}^+$

denote the positive chamber of $\mathfrak{i}'_M(\mathfrak{z})$ with respect to $\Phi^+(\mathfrak{z}_C, \mathfrak{i}_{MC}) = \Phi^+(\mathfrak{g}_C, \mathfrak{i}_C) \cap \Phi(\mathfrak{z}_C, \mathfrak{i}_{MC})$. Write $A^+ = \exp(\mathfrak{a}^+)$. Let $W_M = W(M^0, H_K^0)$, $W_Z = W({}^{k^{-1}}Z, H_K^0)$. Then using (2.2), (3.5), (3.6), (3.7) (2.3) and (3.9),

$$\begin{aligned}
I(\gamma_1, \gamma_2) &= \int_{J_K^0 \times A \times H_p} \varepsilon_R^{J_M}(\gamma_2 j_k h_1) F_f^J(\gamma_1\gamma_2 j_k h_1 h_2)\chi(\gamma_1) h_2^{\gamma(-1)^\nu} \\
&\quad \times \sum_{t \in W_Z \backslash W_M} \det t \xi_{k^{-1}(t\lambda)}(\gamma_2 j_k) \sum_{s \in W(Z, J_M)} \det s c_{\mathfrak{z}}(s : t\lambda : \mathfrak{a}^*) \\
&\quad \times \exp\left(s^{yk^{-1}}(t\lambda)(\log h_1)\right) dj_k\, dh_1\, dh_2 \\
&= [W(Z, J_M)] \sum_{t \in W_Z \backslash W_M} \det t \int_{J_K^0 \times A \times H_p} \varepsilon_R^{J_M}(\gamma_2 j_k h_1) F_f^J(j) \\
&\quad \times \chi(\gamma_1) h_2^{\gamma(-1)^\nu} \xi_{k^{-1}(t\lambda)}(\gamma_2 j_k) \\
&\quad \times c_{\mathfrak{z}}(I : t\lambda : \mathfrak{a}^*) \exp\left(s^{yk^{-1}}(t\lambda)(\log h_1)\right) dj_k\, dh_1\, dh_2 \\
&= [W(Z, J_M)] \sum_{t \in W_Z \backslash W_M} \det t \sum_{s \in W(Z, J_M)} \det s \\
&\quad \times \int_{J_K^0 \times A^+ \times H_p} F_f^J(\gamma_1\gamma_2 j_k h_1 h_2)\chi(\gamma_1) h_2^{\gamma(-1)^\nu} \xi_{k^{-1}(t\lambda)}(\gamma_2 j_k) \\
&\quad \times c_{\mathfrak{z}}(s : t\lambda : \mathfrak{a}^+) \exp\left(s^{yk^{-1}}(t\lambda)(\log h_1)\right) dj_k\, dh_1\, dh_2.
\end{aligned}$$

By definition,

$$^{yk^{-1}}W_Z = {}^{yk^{-1}}W\left({}^{k^{-1}}Z, H_K^0\right) \subseteq W\left(\mathfrak{z}_C, {}^{yk^{-1}}\mathfrak{h}_{kC}\right) = W(Z, J_M).$$

Thus using (2.4) we can write

$$\begin{aligned}
I(\gamma_1, \gamma_2) = \sum_{t \in W_M} \det t \int_{J_K^0 \times A^+ \times H_p} & F_f^J(\gamma_1\gamma_2 j_k h_1 h_2) h_2^{\gamma(-1)^\nu} \chi(\gamma_1) \\
& \times \xi_{k^{-1}(t\lambda)}(\gamma_2 j_k) s(\gamma_2 : t\lambda : h_1)\, dj_k\, dh_1\, dh_2 \qquad (4.4)
\end{aligned}$$

where

$$\begin{aligned}
s(\gamma_2 : \lambda : h_1) = [W(Z, J_M)] \sum_{s \in W(Z, J_M)/{}^{yk^{-1}}W_Z} & \det s c_{\mathfrak{z}}(s : \lambda : \mathfrak{a}^+) \\
& \times \exp\left(s^{yk^{-1}}\lambda(\log h_1)\right).
\end{aligned}$$

Fix $M > 0$. L_χ is $W(M^0, H_K^0)$-stable. If $t \in W(M^0, H_K^0)$, write $t \cdot L_\chi^M = \{t\lambda | \lambda \in L_\chi^M\}$. Then

$$\sum_{\chi\in Z(\mathfrak{h}_p)^{\wedge}} \sum_{\lambda\in L_\chi^M} \overline{(\chi\otimes\xi_\lambda)(h_k)} \int_{\mathfrak{h}_p^*} h_p^{-\sqrt{(-1)}\nu}$$

$$\times \int_{G^J} \theta\big(H, \lambda, \chi\otimes\xi_H^{-1}, \nu\big)(x) f(x) d_G(x)\, d\nu$$

$$= \frac{(-1)^{r_I(J)}[M/M^+]}{[W(M, J_M)]} \sum_{t\in W(M^0, H_K^0)} \det t$$

$$\times \sum_{\gamma_2\in\Gamma_2} \sum_{\gamma_1\in\Gamma_1} \sum_{\chi\in Z(\mathfrak{h}_p)^{\wedge}} \sum_{\lambda\in t^{-1}L_\chi^M} \overline{(\chi\otimes\xi_\lambda)(t^{-1}h_k)}$$

$$\times \int_{\mathfrak{h}_p^*} h_p^{-\sqrt{(-1)}\nu} \int_{J_K^0\times A^+\times H_p} F_f^J(\gamma_1\gamma_2 j_k h_1 h_2) h_2^{\sqrt{(-1)}\nu}$$

$$\times \chi(\gamma_1)\xi_{k^{-1}(t\lambda)}(\gamma_2 j_k) s(\gamma_2 : \lambda : h_1)\, dj_k\, dh_1\, dh_2. \tag{4.5}$$

Fix $\gamma_2 \in \Gamma_2$ and use the notation of (2.2) for $J_I^+ = \gamma_2 J_K^0$. Then ${}^{k^{-1}}H_K^0 = J_K^0 \exp({}^{y^{-1}}\sqrt{-1}\,\mathfrak{a})$. Write ${}^{k^{-1}}H_k^0 = T$, $J_K^0 = T_1$ and $\exp({}^{y^{-1}}\sqrt{-1}\,\mathfrak{a}) = T_2$. Let $L^* = \{\lambda \in {}^{k^{-1}}L_H | {}^{y}\lambda|\mathfrak{a} = 0 \text{ and } \xi_{k\lambda}|_{\Gamma_0} = 1\}$. Let $L_p = \{\sum_{i=1}^{l} n_i\alpha_i | n_i \in Z,\ 1 \le i \le l\}$ where $\{\alpha_1, \ldots, \alpha_l\}$ is a set of simple roots for $\Phi(\mathfrak{z}_C, \mathfrak{j}_{MC})$. Let $L_0 = L^* \oplus {}^{y^{-1}}L_p$. L^* and ${}^{y^{-1}}L_p$ parameterize the character groups of $T_1/T_1 \cap \Gamma_0 T_2$ and $T_2/T_2 \cap C(Z)$ respectively, where $C(Z)$ denotes the center of Z. ${}^{k^{-1}}L/L_0$ has finite order. Pick representatives $\tau_1, \ldots, \tau_m \in {}^{k^{-1}}L$ of the cosets ${}^{k^{-1}}L/L_0$. Then every element λ_0 of L_H can be written uniquely as $\lambda_0 = {}^{k}(\tau^* + {}^{y^{-1}}\lambda + \tau_i)$, $\tau^* \in L^*$, $\lambda \in L_p$, $1 \le i \le m$, and the consistency condition $\xi_{\lambda_0}|_{\Gamma_0} = \chi|_{\Gamma_0}$ is equivalent to $\xi_{k\tau_i}|_{\Gamma_0} = \chi|_{\Gamma_0}$. Let $Z(\mathfrak{h}_p)_i^{\wedge} = \{\chi \in Z(\mathfrak{h}_p)^{\wedge} | \chi|_{\Gamma_0} = \xi_{k\tau_i}|_{\Gamma_0}\}$.

Write $h_k = \gamma_0 h_k^0$, $\gamma_0 \in Z(\mathfrak{h}_p)$, $h_k^0 \in H_K^0$. For $t \in W(M^0, H_K^0)$, write ${}^{k^{-1}}(t^{-1}h_k^0) = j_1(t) j_2(t)$, where $j_1(t) \in T_1$, $j_2(t) = \exp({}^{y^{-1}}(-\sqrt{-1}\, J_2(t))) \in T_2$, $J_2(t) \in \mathfrak{a}$. Using the above decompositions of λ_0 and $t^{-1}h_k$, we have:

$$\overline{\chi\otimes\xi_{\lambda_0}(t^{-1}h_k)} = \chi(\gamma_0)\, \overline{\xi_{\tau^*+\tau_i}(j_1(t)) e}^{\sqrt{(-1)}(\lambda + {}^{y}\tau_i)(J_2(t))}; \tag{4.6}$$

$$\xi_{k^{-1}\lambda_0}(\gamma_2 j_k) = \xi_{y\tau_i}(\gamma_2)\xi_{\tau^*+\tau_i}(j_k); \tag{4.7}$$

$$s(\gamma_2 : \lambda_0 : h_1) = s\big(\gamma_2 : {}^{ky^{-1}}\lambda + {}^{k}\tau_i : h_1\big). \tag{4.8}$$

Thus for fixed $t \in W(M^0, H_K^0)$, $\gamma_2 \in \Gamma_2$, with the understanding that all sums are restricted to $t^{-1}L_H^M$,

$$\sum_{\chi\in Z(\mathfrak{h}_p)^\wedge}\ \sum_{\lambda_0\in t^{-1}L_\chi^M}\overline{(\chi\otimes\xi_{\lambda_0})(t^{-1}h_k)}\int_{\mathfrak{h}_p^*}h_p^{-\sqrt{(-1)}\nu}$$

$$\times\sum_{\gamma_1\in\Gamma_1}\int_{J_K^0\times A^+\times H_p}F_f^J(\gamma_1\gamma_2h_1h_2)\,h_2^{\sqrt{(-1)}\nu}\chi(\gamma_1)$$

$$\times\ \xi_{k^{-1}(\lambda)}(\gamma_2j_k)s(\gamma_2:\lambda:h_1)\,dj_k\,dh_1\,dh_2$$

$$=\sum_{i=1}^m\xi_{y_{\tau_i}}(\gamma_2)\int_{\mathfrak{h}_p^*}h_p^{-\sqrt{(-1)}\nu}\int_{A^+}\sum_{\lambda\in L_p}e^{\sqrt{(-1)}(\lambda+{}^{y}\tau_i)(J_2(t))}s\left(\gamma_2:{}^{ky^{-1}}\lambda+{}^{k}\tau_i:h_1\right)$$

$$\times\int_{H_p}h_2^{\sqrt{(-1)}\nu}\sum_{\chi\in Z(\mathfrak{h}_p)_i^\wedge}\chi(\gamma_0)\sum_{\tau^*\in L^*}\overline{\xi_{\tau^*+\tau_i}(j_1(t))}\sum_{\gamma_1\in\Gamma_1}\chi(\gamma_1)$$

$$\times\int_{J_K^0}\xi_{\tau^*+\tau_i}(j_k)F_f^J(\gamma_1\gamma_2j_kh_1h_2)\,dj_k\,dh_1\,dh_2\,d\nu. \tag{4.9}$$

LEMMA 4.10. *Fix $i=1,\ldots,m$. Then, if the sum is taken over all of L^*,*

$$\sum_{\chi\in Z(\mathfrak{h}_p)_i^\wedge}\chi(\gamma_0)\sum_{\tau^*\in L^*}\overline{\xi_{\tau^*+\tau_i}(j_1(t))}\sum_{\gamma_1\in\Gamma_1}\chi(\gamma_1)\int_{J_K^0}\xi_{\tau^*+\tau_i}(j_k)F_f^J(\gamma_1\gamma_2j_kh_1h_2)\,dj_k$$

$$=\frac{(\operatorname{vol}J_K^0)[\Gamma_1]}{[\Gamma_4]}\sum_{\gamma_4\in\Gamma_4}\xi_{\tau_i}(\gamma_4)F_f^J(\gamma_2\gamma_4\gamma_0j_1(t)h_1h_2),$$

where $\Gamma_4=Z(\mathfrak{h}_p)J_K^0\cap\Gamma_0Z(\mathfrak{a})$. The sum over L^* converges uniformly for $h_1h_2\in AH_p$ and uniformly for compacta of $j_1(t)\in(J_K^0)'$, but not absolutely.

PROOF. Fix $h_1h_2\in AH_p$, and $\chi_i\in Z(\mathfrak{h}_p)_i^\wedge$. For $j_k\in Z(\mathfrak{h}_p)J_K^0$, define

$$g(j_k)=\sum_{\gamma_4\in\Gamma_4}(\chi_i\otimes\xi_{\tau_i})(\gamma j_k)F_f^J(\gamma_2\gamma j_kh_1h_2).$$

Then g is well defined on $Z(\mathfrak{h}_p)J_K^0/\Gamma_4$, and by elementary Fourier analysis,

$$\sum_{\chi\in Z(\mathfrak{h}_p)_i}\chi(\gamma_0)\sum_{\tau^*\in L^*}\overline{\xi_{\tau^*+\tau_i}(j_1(t))}\sum_{\gamma_1\in\Gamma_1}\chi(\gamma_1)\int_{J_K^0}\xi_{\tau^*+\tau_i}(j_k)F_f^J(\gamma_1\gamma_2j_kh_1h_2)\,dj_k$$

$$=\frac{1}{[\Gamma_4]}\overline{(\chi_i\otimes\xi_{\tau_i})(\gamma_0j_1(t))}\sum_{\chi\in(Z(\mathfrak{h}_p)/\Gamma_0)^\wedge}\chi(\gamma_0)$$

$$\times\sum_{\tau^*\in L^*}\overline{\xi_{\tau^*}(j_1(t))}\int_{Z(\mathfrak{h}_p)J_K^0/\Gamma_4}(\chi\otimes\xi_{\tau^*})(j_k)g(j_k)\,d\bar{j}_k$$

$$=\frac{1}{[\Gamma_4]}\overline{(\chi_i\otimes\xi_{\tau_i})(\gamma_0j_1(t))}\sum_{b^*\in(Z(\mathfrak{h}_p)J_K^0/\Gamma_4)^\wedge}\overline{b^*(\gamma_0j_1(t))}$$

$$\times \int_{Z(\mathfrak{h}_p)J_K^0/\Gamma_4} b^*(j_k)\, g(j_k)\, d\bar{j}_k$$

$$= \frac{[\Gamma_1](\mathrm{vol}\, J_K^0)}{[\Gamma_4]} \overline{\chi_i \otimes \xi_{\tau_i}(\gamma_0 j_1(t))}\, g(\gamma_0 j_1(t))$$

$$= \frac{[\Gamma_1](\mathrm{vol}\, J_K^0)}{[\Gamma_4]} \sum_{\gamma_4 \in \Gamma_4} \xi_{\tau_i}(\gamma_4) F_f^J(\gamma_2\gamma_1\gamma_0 j_1(t) h_1 h_2).$$

Here $d\bar{j}_k$ on $Z(\mathfrak{h}_p)J_K^0/\Gamma_4$ is defined by

$$\sum_{\gamma_1 \in \Gamma_1} \int_{J_K^0} f(\gamma_1 j_k)\, dj_k = \frac{1}{[\Gamma_4]} \sum_{\gamma_4 \in \Gamma_4} \int_{Z(\mathfrak{h}_p)J_K^0/\Gamma_4} f(\gamma_4 j_k)\, d\bar{j}_k$$

for $f \in C(Z(\mathfrak{h}_p)J_K^0)$. Q.E.D.

LEMMA 4.11. *There is a dense open set T_2^* of T_2 such that*

$$\left| \sum_{\substack{\lambda \in L_p \ni \\ {}^{y^{-1}}\lambda + \tau_i \in {}^{k^{-1}}(t^{-1} \cdot L_H^M)}} e^{\sqrt{(-1)}(\lambda + {}^y\tau_i)(J_2(t))} s\left(\gamma_2 : {}^{ky^{-1}}\lambda + {}^k\tau_i : h_1\right) \right|$$

is bounded independent of M, uniformly for $h_1 \in A^+$, and for $j_2(t) = {}^y(\exp - \sqrt{-1}\, J_2(t))$ in compact subsets of T_2^*.

PROOF. $L_p \subseteq \mathfrak{i}_M^* = F_2$. Let $F_2' = \{\lambda \in F_2 | \langle \alpha, \lambda \rangle \neq 0,\ \alpha \in \Phi(\mathfrak{z}_\mathbf{C}, \mathfrak{i}_{M\mathbf{C}})\}$. Let $C_1, \ldots, C_k$ be the chambers of F_2' for which $C_j \subseteq F_- = \{\lambda \in F_2 | \lambda(H) < 0$ for all $H \in \bar{\mathfrak{a}}^+$, $H \neq 0\}$, $j = 1, \ldots, k$. These are the chambers of F_2' on which $c_\delta(s: {}^{ky^{-1}}(s^{-1}\lambda) : \mathfrak{a}^+) = c(s : s^{-1}C_j)$ is constant, $\lambda \in C_j$, and possibly nonzero. In addition, we need to consider a finite number of "singular chambers" of the form $C_{\lambda_0} = \{\lambda \in F_2 | \lambda \in \bar{C}^+$ if and only if $\lambda_0 \in \bar{C}^+$ for every chamber C^+ of $F_2'\}$, for fixed singular $\lambda_0 \in L_p$. Let $C_{k+1}, \ldots, C_r$ be the distinct "singular chambers" for which $C_j \subseteq F_-, j = k+1, \ldots, r$. Then $c_\delta(s : {}^{ky^{-1}}(s^{-1}\lambda) : \mathfrak{a}^+) = c(s : s^{-1}C_j)$ is constant for $\lambda \in C_j$, $k + 1 < j < r$. Further, $c_\delta(s : {}^{ky^{-1}}(s^{-1}\lambda) : \mathfrak{a}^+) = 0$ if $\lambda \notin \bigcup_{j=1}^r C_j$. Then, all sums being restricted to ${}^{y^{-1}}\lambda + \tau_i \in {}^k(t^{-1} \cdot L_H^M)$,

$$\sum_{\lambda \in L_p} e^{\sqrt{(-1)}(\lambda + {}^y\tau_i)(J_2(t))} \sum_{s \in W(Z, J_M)/{}^{yk^{-1}}W_Z} \det s c_\delta\left(s : {}^{ky^{-1}}\lambda + {}^k\tau_i : \mathfrak{a}^+\right)$$

$$\times \exp(s(\lambda + {}^y\tau_i)(\log h_1))$$

$$= \sum_{s \in W(Z, J_M)/{}^{yk^{-1}}W_Z} \det s \sum_{i=1}^{r} c\left(s : s^{-1}C_j\right) \sum_{\substack{\lambda \in L_p \ni \\ \lambda + {}^y\tau_i \in C_j}} e^{(\lambda + {}^y\tau_i)(\sqrt{(-1)}s^{-1}J_2(t) + \log h_1)}.$$

Fix $s \in W(Z, J_M)$ and $1 \leqslant j \leqslant r$. Write $\log h_1 + \sqrt{-1}\, s^{-1}J_2(t) = H$. Then

$$\sum_{\substack{\lambda \in L_p \ni \\ \lambda + {}^{y}\tau_i \in C_j}} e^{(\lambda + {}^{y}\tau_i)(H)} = e^{{}^{y}\tau_i(H)} \sum_{\substack{\lambda \in L_p \ni \\ \lambda + {}^{y}\tau_i \in C_j}} e^{\lambda(H)}$$

is just a geometric sum over a lattice in a convex subset of F_2, and its absolute value can be bounded by a finite number of terms (depending on C_j, but not on M) of the form

$$\frac{2|e^{\lambda_0(H)}|}{|1 - e^{\lambda_1(H)}| \cdots |1 - e^{\lambda_s(H)}|}$$

where $\lambda_0 \in C_j \cap F_-$ and where λ_i, $1 \leqslant i \leqslant s \leqslant l$, are integral linear combinations of $\alpha_1, \ldots, \alpha_l$. Let $\Lambda(J, \gamma_2, t)$ denote the finite set of λ_i which arise in the denominators obtained from the geometric sums over C_j, $1 \leqslant j \leqslant r$. Define $T_2^*(J, \gamma_2, t) = \{e^{-\sqrt{(-1)}yH} \in T_2 | H \in \mathfrak{a},\ e^{\sqrt{(-1)}\lambda(s^{-1}H)} \neq 1,\ s \in W(Z, J_M),\ \lambda \in \Lambda(J, \gamma_2, t)\}$. Then if $|1 - \cos \lambda(s^{-1}J_2(t))| > \varepsilon$ for all $s \in W(Z, J_M)$, $\lambda \in \Lambda(J, \gamma_2, t)$, we have

$$\frac{2|e^{\lambda_0(H)}|}{|1 - e^{\lambda_1(H)}| \cdots |1 - e^{\lambda_s(H)}|} \leqslant \frac{2}{\varepsilon^s}$$

since $\mathrm{Re}(\lambda_0(H)) = \lambda_0(\log h_1) < 0$, $\lambda_0 \in F_-$, $h_1 \in A^+$. Q.E.D.

Let $H^* = \{h \in H' |$ for every $J \in \mathrm{Car}'(L, H)$, $t \in W(M^0, H_K^0)$, $\gamma_2 \in \Gamma_2$, in the decomposition ${}^{k^{-1}}(t^{-1}h_k^0) = j_1(t)j_2(t)$, $j_1(t) \in T_1$, $j_2(t) \in T_2$ used in (4.9), we have $j_2(t) \in T_2^*(J, \gamma_2, t)\}$.

LEMMA 4.12.

$$I_f^H(h_0) = \frac{(-1)^{r_I(H)+1}[M/M^+]}{\mathrm{vol}\, H_K (2\pi)^{d(H_p)}} \sum_{J \in \mathrm{Car}'(L, H)} \frac{(-1)^{r_I(J)}}{[W(M, J_M)]}$$
$$\times \sum_{t \in W(M^0, H_K^0)} \det t \sum_{\gamma_2 \in Z(\mathfrak{a})\Gamma_0} I_f^H(J, t, \gamma_2, h_0)$$

where

$$I_f^H(J, t, \gamma_2, h_0)$$
$$= \int_{\mathfrak{h}_p^*} h_p^{-\sqrt{(-1)}\nu} \int_{A^+} \sum_{i=1}^{m} \xi_{\tau_i}(\gamma_2) \sum_{\lambda \in L_p} e^{\sqrt{(-1)}(\lambda + {}^{y}\tau_i)(J_2(t))} s\left(\gamma_2 : {}^{ky^{-1}}\lambda + {}^{k}\tau_i : h_1\right)$$
$$\times \int_{H_p} h_2^{\sqrt{(-1)}\nu} \frac{(\mathrm{vol}\, J_K^0)[\Gamma_1]}{[\Gamma_4]} F_f^J(\gamma_2\gamma_0 j_1(t) h_1 h_2)\, dh_2\, dh_1\, d\nu.$$

PROOF. By Lemmas 4.10 and 4.11, the sums over L^* and L_p are bounded independent of M, uniformly in ν, h_1, and h_2. Thus the $\lim_{M\to\infty}$ can be taken inside of each of the integrals. The sums over $\Gamma_2 = Z(\mathfrak{a})/Z(\mathfrak{a}) \cap Z(\mathfrak{h}_p)J_K^0$ and $\Gamma_4 = \Gamma_0 Z(\mathfrak{a}) \cap Z(\mathfrak{h}_p)J_K^0 = \Gamma_0(Z(\mathfrak{a}) \cap Z(\mathfrak{h}_p)J_K^0)$ have been combined to yield a sum over $Z(\mathfrak{a})\Gamma_0$. Q.E.D.

Using (3.3), (3.4), we have that for $h_1 \in A, j_k \in J_K$,

$$\int_{H_p} h_2^{\sqrt{(-1)}\nu} F_f^J (j_k h_1 h_2)\, dh_2$$
$$= (2\pi)\frac{(d(H_p) - d(A))/2}{\operatorname{vol} J_K} \sum_{b^* \in \hat{J}_K} \overline{b^*(j_k)} \int_{\mathfrak{a}^*} h_1^{-\sqrt{(-1)}\mu} \hat{F}_f^J (b^*, \mu \otimes \nu)\, d\mu \tag{4.13}$$

where the sum over $b^* \in \hat{J}_K$ converges uniformly on compacta of J_K', but not absolutely, and the integral over $\mathfrak{a}^*$ converges absolutely. The integral

$$I_J^H (\gamma_2 : J_2(t), \mu) = \int_{A^+} \sum_{i=1}^{m} \xi_{\tau_i}(\gamma_2) \sum_{\lambda \in L_p} e^{\sqrt{(-1)}(\lambda + \nu_{\tau_i})(J_2(t))}$$
$$\times s(\gamma_2 : {}^{k\gamma^{-1}}\lambda + {}^{k}\tau_i : h_1) h_1^{-\sqrt{(-1)}\mu}\, dh_1$$

converges absolutely also, since the absolute value of the integrand is bounded by a finite number of terms of the form $ce^{\lambda_0(\log h_1)}$, c a constant (varying uniformly with $e^{-\sqrt{(-1)}\nu J_2(t)}$ in compacta of T_2^*) and $\lambda_0 \in F_-$. (Here

$$\int_{A^+} e^{\lambda_0(\log h_1)}\, dh_1 = c_A \int_0^\infty \cdots \int_0^\infty e^{-n_1 r_1 - \cdots - n_l r_l}\, dr_1 \cdots dr_l$$

where $n_i > 0$ for $1 < i < l$, c_A a constant relating dh_1 to $dr_1 \cdots dr_l$.) Thus we may use (4.13) and then exchange the order of integration to obtain:

$$I_f^H(J, t, \gamma_2, h_0) = (2\pi)\frac{(d(H_p) - d(A))/2}{[Z(\mathfrak{a}) \cdot \Gamma_0][Z(\mathfrak{a}) \cap Z(\mathfrak{h}_p)]}$$
$$\times \int_{\mathfrak{h}_p^*} h_p^{-\sqrt{(-1)}\nu} \int_{\mathfrak{a}^*} I_J^H(\gamma_2 : J_2(t), \mu)$$
$$\times \sum_{b^* \in \hat{J}_K} \overline{b^*(\gamma_0\gamma_2 j_1(t))}\, \hat{F}_f^J (b^*, \mu \otimes \nu)\, d\mu\, d\nu. \tag{4.14}$$

THEOREM 4.15. *Let* $h_0 = h_k h_p \in H^*$. *Then*

$$F_f^H(h_0) = \frac{(-1)^{r_I(H)}}{\operatorname{vol} H_K (2\pi)^{\dim(h_p)}} \sum_{b^* \in \hat{H}_K} \overline{b^*(h_k)} \int_{\mathfrak{h}_p^*} h_p^{-\sqrt{(-1)}\nu} \theta(H, b^*, \nu)(f)\, d\nu$$

$$+ \frac{(-1)^{r_I(H)+1}[M/M^+]}{\operatorname{vol} H_K}$$

$$\times \sum_{J \in \operatorname{Car}'(L, H)} \frac{1}{(2\pi)^{\dim \mathfrak{i}_p}[W(M, J_M)][Z(\mathfrak{a}) \cdot \Gamma_0][Z(\mathfrak{a}) \cap Z(\mathfrak{h}_p)]}$$

$$\times \sum_{t \in W(M^0, H_K^0)} \det t \sum_{\gamma_2 \in Z(\mathfrak{a})\Gamma_0}$$

$$\times \Bigg[\sum_{b^* \in \hat{J}_K} \overline{b^*(\gamma_0 \gamma_2 j_1(t))} \int_{\mathfrak{h}_p^*} h_p^{-\sqrt{(-1)}\nu} \int_{\mathfrak{a}^*} I_J^H(\gamma_2 : J_2(t) : \mu)$$

$$\times \theta(J, b^*, \mu \otimes \nu)(f)\, d\mu\, d\nu + I_J^H(f, h_0, t, \gamma_2) \Bigg]$$

where

$$I_J^H(f, h_0, t, \gamma_2) = -\int_{\mathfrak{h}_p^*} h_p^{-\sqrt{(-1)}\nu} \int_{\mathfrak{a}^*} I_J^H(\gamma_2 : J_2(t) : \mu) \sum_{b^* \in \hat{J}_K} \overline{b^*(\gamma_0 \gamma_2 j_1(t))}$$

$$\times \left\{ \sum_{A \in \operatorname{Car}'(G, J)} \int_{G^A} \theta(J, b^*, \mu \otimes \nu)(x) d_G(x) \right\} d\mu\, d\nu.$$

PROOF. The theorem follows from combining Lemmas 4.2 and 4.12 along with (4.14) and then using Lemma 4.1 and J in place of H. It also uses the fact that

$$\sum_{b^* \in \hat{J}_K} \overline{b^*(j_k)} \int_{\mathfrak{h}_p^*} h_p^{-\sqrt{(-1)}\nu} \int_{a^*} I_J^H(\gamma_2 : J_2(t) : \mu) \theta(J, b^*, \mu \otimes \nu)(f)\, d\mu\, d\nu$$

converges absolutely.

5. Proof of Lemma 2.5. It suffices to prove the lemma with $s = I$ because of (2.3). If $\lambda_0 \in F^+ \cap L_{\mathfrak{z}}$ is regular, and $t_i(\lambda_0) = 0$, then there are elements $\lambda' \in F^+ \cap L_{\mathfrak{z}}$ with $t_i(\lambda') > 0$, and so $c(I : \lambda_0) = c(I : \lambda') = 0$. If $\lambda_0 \in L_{\mathfrak{z}}^s$, let F^+ be a component of $F'_{\mathfrak{z}}$ whose closure contains λ_0. Then $\{wF^+ | w \in W'\}$ are exactly the components of F' whose closures contain λ_0, and so $c(I : \lambda_0) = (1/[W'])\sum_{w \in W'} c(I : wF^+)$.

Suppose first that there is a root $\alpha \in \Phi(\mathfrak{z}_C, \mathfrak{j}_C)$ such that ${}^{ky^{-1}}\alpha$ is a compact root of $(\mathfrak{z}, \mathfrak{h}_k)$ and $\langle \lambda_0, \alpha \rangle = 0$. Then $s_\alpha \in W'$, where s_α denotes the reflection corresponding to α, and

$$\sum_{w \in W'} \det w\, c(w : \lambda_0) = \frac{1}{[W']} \sum_{w \in W'} \det w \sum_{v \in W'} c(w : w^{-1}vF^+)$$

$$= \frac{1}{[W']} \sum_{v \in W'} \sum_{w \in W'/\{I, s_\alpha\}} \det w\big(c(w : w^{-1}vF^+) - c(ws_\alpha : s_\alpha w^{-1}vF^+)\big) = 0$$

using (2.4), since $s_\alpha \in W(Z, {}^{yk^{-1}}H_K)$.

Thus we may assume that

$${}^{ky^{-1}}\Phi^+(\lambda_0) = {}^{ky^{-1}}\{\alpha \in \Phi^+ | \langle \lambda_0, \alpha \rangle = 0\}$$

consists entirely of singular imaginary roots of $(\mathfrak{z}, \mathfrak{h}_k)$. As a consequence, if β_1, $\beta_2 \in \Phi^+(\lambda_0)$, then $\langle \beta_1, \beta_2 \rangle = 0$ and no other element of Φ^+ is in the linear span of β_1 and β_2. For $\alpha \in \Phi^+$, let $H_\alpha = \{\lambda \in F_\mathfrak{z} | \langle \alpha, \lambda \rangle = 0\}$.Then if $v \in W'$ and $\alpha \in \Phi^+(\lambda_0)$, H_α is the only root hyperplane separating vF^+ and $s_\alpha vF^+$.

Let $F_- = \{\lambda \in F_\mathfrak{z} | t_i(\lambda) < 0,\ i = 1, \ldots, l\}$. Suppose $w_1, \ldots, w_r$ are the elements of W' satisfying $F_i = w_i F^+ \subseteq F_-$, $1 \leqslant i \leqslant r$. $\overline{F_1 \cup \cdots \cup F_r}$ is a convex set, bounded by certain hyperplanes H_α, $\alpha \in \Phi(\mathfrak{z}_C, \mathfrak{j}_C)$. Since λ_0 is not in the interior of $\overline{F_1 \cup \cdots \cup F_r}$ which is contained in F^-, λ_0 is in at least one hyperplane H_{α_0} bounding $\overline{F_1 \cup \cdots \cup F_r}$. Then $s_0 = s_{\alpha_0} \in W'$ and $s_0 F_i \not\subseteq F_-$, $i = 1, \ldots, r$. Thus

$$\sum_{w \in W'} \det w\, c(w : \lambda_0) = \frac{1}{[W']} \sum_{w \in W'} \det w \sum_{v \in W'} c(w : w^{-1}vF^+)$$

$$= \frac{1}{[W']} \sum_{w \in \{I, s_0\}\backslash W'} \det w$$

$$\times \sum_{v \in \{I, s_0\}\backslash W'} \big[c(w : w^{-1}vF^+) + c(w : w^{-1}s_0vF^+) - c(s_0w : w^{-1}s_0vF^+) - c(s_0w : w^{-1}vF^+)\big].$$

If neither vF^+ nor s_0vF^+ is contained in F_-, the quantity in brackets is zero. Thus we may assume $vF^+ \subseteq F_-$. Then $s_0vF^+ \not\subseteq F_-$ and so $c(w : w^{-1}s_0vF^+) = c(s_0w : w^{-1}vF^+) = 0$.

Write $s_0 = s_1 \cdots s_k$ as a product of simple reflections with k minimal, s_i the reflection corresponding to $\alpha_{l(i)}$, $1 \leqslant l(i) \leqslant l$. Note that k will be an odd integer. Let $\Gamma_i \in \mathfrak{j}_p^+$ be a semiregular element of $\mathfrak{j}$ corresponding to $\alpha_{l(i)}$, $\mathfrak{h}_{\Gamma_i}^+ = \mathfrak{j}$ and $\mathfrak{h}_{\Gamma_i}^- = \mathfrak{j}_i$ the corresponding Cartan subalgebras of $\mathfrak{z}$ [8, §1.3.4].

Let $\mathfrak{z}_i$ be the centralizer in $\mathfrak{z}$ of $(\mathfrak{j}_i)_k$. Let $\mathfrak{j}_i^+$ be the chamber of $\mathfrak{j}_i^+(\mathfrak{z}_i)$ containing Γ_i. Write $c_i(s : uF^+)$ for $c_{\mathfrak{z}_i}(s : uF^+ : \mathfrak{j}_i^+)$, $s, u \in W(\mathfrak{z}_C, \mathfrak{j}_C)$.

It follows from [1(d)] that

$$c(w : w^{-1}vF^{+}) = \sum_{i=1}^{k} (-1)^{i+1}\Big(c_i\big(s_{i+1} \cdots s_k w : w^{-1}vF^{+}\big) + c_i\big(s_i s_{i+1} \cdots s_k w : w^{-1}vF^{+}\big)\Big)$$

and

$$c\big(s_0 w : w^{-1}s_0 vF^{+}\big) = \sum_{i=1}^{k} (-1)^{i+1}\Big(c_i\big(s_i \cdots s_k w : w^{-1}s_0 vF^{+}\big) + c_i\big(s_{i+1} \cdots s_k w : w^{-1}vF^{+}\big)\Big).$$

We claim that for each $i = 1, \ldots, k$,

$$c_i\big(s_{i+1} \cdots s_k w : w^{-1}vF^{+}\big) = c_i\big(s_{i+1} \cdots s_k w : w^{-1}s_0 vF^{+}\big)$$

and

$$c_i\big(s_i \cdots s_k w : w^{-1}vF^{+}\big) = c_i\big(s_i \cdots s_k w : w^{-1}s_0 vF^{+}\big)$$

so that $c(w : w^{-1}vF^{+}) = c(s_0 w : w^{-1}s_0 vF^{+})$. We will show the first equality. The second follows from a similar argument.

To prove the equality, it suffices to show that no hyperplane H_β, $\beta \in \Phi(\mathfrak{z}_{iC}, \mathfrak{j}_C)$ separates $s_{k+1} \cdots s_k vF^{+}$ and $s_{i+1} \cdots s_k s_0 vF^{+} = s_i \cdots s_1 vF^{+}$. Suppose such a root β exists. Let $\beta' = s_k \cdots s_{i+1}\beta$. Then $H_{\beta'}$ separates vF^{+} and $s_0 vF^{+}$. But H_{α_0} is the only hyperplane separating vF^{+} and $s_0 vF^{+}$, so that $\alpha_0 = \beta'$, and $\beta = s_{i+1} \cdots s_k \alpha_0$. Since $\beta \in \Phi((\mathfrak{z}_i)_C, \mathfrak{j}_C)$, $\langle \beta, \alpha_{l(i)} \rangle = 0$ and $\beta = s_{i+1} \cdots s_k \alpha_0 = s_i s_{i+1} \cdots s_k \alpha_0$. But since $s_1 \cdots s_k$ is a reduced decomposition of s_0 into simple reflections, this is impossible. Q.E.D.

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